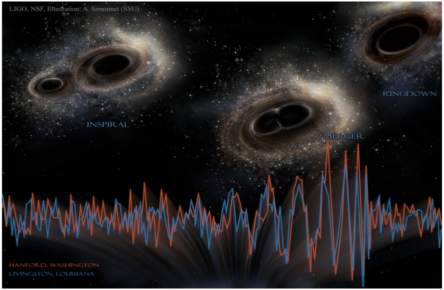
UNIVERSITY OF CAMBRIDGE

Listening to Ripples in Space-Time: Cosmological Inference from Standard Sirens

Jacob Tutt[1]

[1] Department of Physics, University of Cambridge, UK

New Era of GW Detection



Credit: A. Simmonet (Sonoma State)

Scientific Opportunities

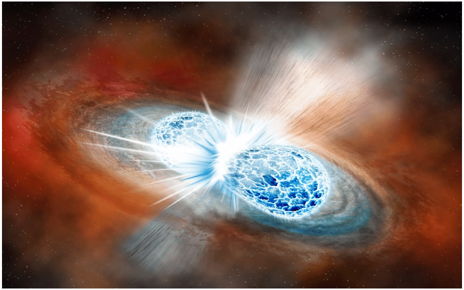
- Probe extreme matter and fields
- Enable multi-messenger astronomy (GW170817 - 2017)
- ‘Standard sirens’ allow construction of distance ladder

Gravitational Wave Detections

1916: Einstein's Theoretical Prediction

2025: Over 290 GW Events Detected

2015: First Observational Detection – GW150914



Credit: NASA

Statistical Detection of GWs

Goal:

- Implementation of some of the key statistical approaches used for GW detection and localisation.

- **Approach:**
  - ‘Match Filtering’ for:
    - Signal identification
    - Sky Triangulation
  - A Bayesian approach to GW analysis:
    - Sky Localisation, RA and Dec
    - Luminosity Distance, D_L
  - Cosmological Inference of Hubble’s Constant, H_0

Interferometer Data

Strain and Space-Time Distortion

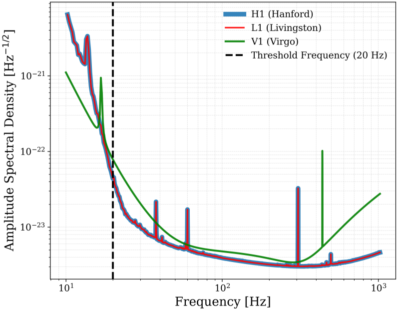
$$h(t) = \frac{\Delta L(t)}{L}$$

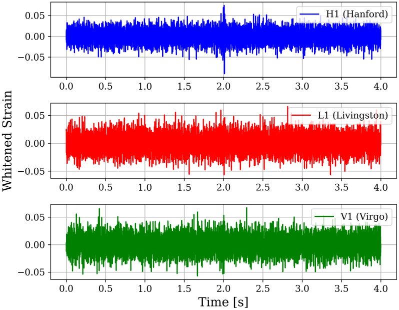
Interferometer Arm Lengths, L:

- **H1 and L1:** 4 km
- **V1:** 3 km

Power Spectral Density (PSD)

$$\langle \tilde{n}(f)\tilde{n}^*(f') \rangle = \frac{1}{2} S_n(f)\delta(f - f')$$





Antenna Response Functions

Antenna Sensitivity:

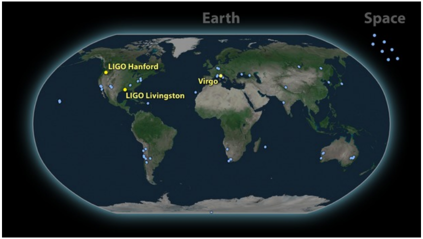
Orientation (3D) of the detector relative to the quadrupolar distortion:

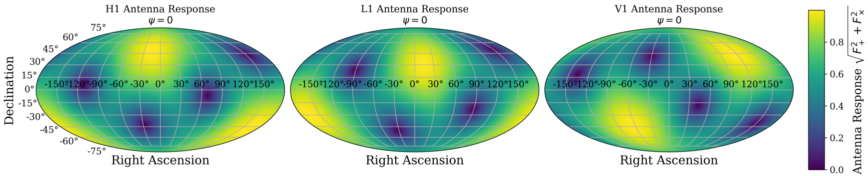
$$\tilde{h}_{\text{det}}(f) = F_+ \tilde{h}_+(f) + F_\times \tilde{h}_\times(f)$$

Where F_+ and $F_\times$ depend on:

- Right ascension and declination
- Polarisation angle Ψ
- GPS Time t_{GPS}

Geography of Interferometers:





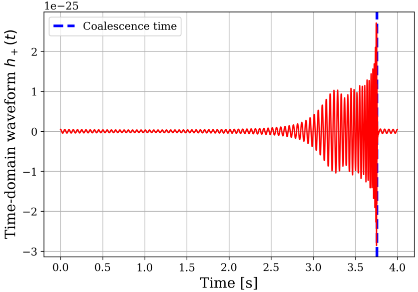
Waveform Models

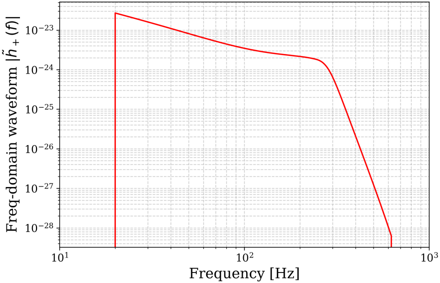
Waveform + Detection Parameters:

$$\boldsymbol{\theta}_{\mathrm{BBH}} = \{m_1, m_2, \vec{S}_1, \vec{S}_2, D_L, \alpha, \delta, \iota, \psi, t_c, \phi_c\} \tag{1}$$

Detection Parameters:

$$\boldsymbol{\theta}_{\mathrm{Red}} = \{D_L, \alpha, \delta, \psi, t_c\} \tag{2}$$





Match Filtering: Mathematical Derivation

A given 'optimal filter', $K(t)$ and the interferometer's detected signal: $s(t) = n(t) + h(t)$ - cross corrolation leads to:

$$\rho_{\mathrm{mf}}^2(t_c) = \frac{S^2}{N^2} = (S \star K)(t_c) = \int_{-\infty}^{\infty} dt\, S(t)K(t - t_c) \tag{3}$$

Parseval Theorem: $$\int_{-\infty}^{\infty} dt\, S(t)K^*(t) = \int_{-\infty}^{\infty} df\, \tilde{S}(f)\tilde{K}^*(f) \tag{4}$$

Fourier Translation: $$\mathcal{F}[K(t-t_c)](f) = \tilde{K}(f)e^{-2\pi i f t_c} \tag{5}$$

Overall: $$\rho_{\mathrm{mf}}^2(t_c) = \int_{-\infty}^{\infty} df\, \tilde{S}(f)\tilde{K}^*(f)e^{2\pi i f t_c} \tag{6}$$

Match Filtering: Mathematical Derivation

$$\rho^2_{\mathrm{mf}}(t_c) = \int_{-\infty}^{\infty} df\, \tilde{S}(f)\tilde{K}^*(f)e^{2\pi i f t_c} \tag{7}$$

Optimal Filter (Cauchy Schwarz): $\tilde{K}(f) = \dfrac{\tilde{h}(f)}{S_n(f)}$ (8)

Real Valued Time Series: $\rho^2_{\mathrm{mf}}(t_c) = 4\Re \displaystyle\int_0^{\infty} df\, \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_n(f)} e^{2\pi i f t_c}$ (9)

Inverse Fourier Tranform: $\rho^2_{\mathrm{mf}}(t_c) = 4\Re\mathcal{F}^{-1}\left[\dfrac{\tilde{s}(f)\tilde{h}^*(f)}{S_n(f)}\right](t_c)$ (10)

Matched Filtering: Implementation

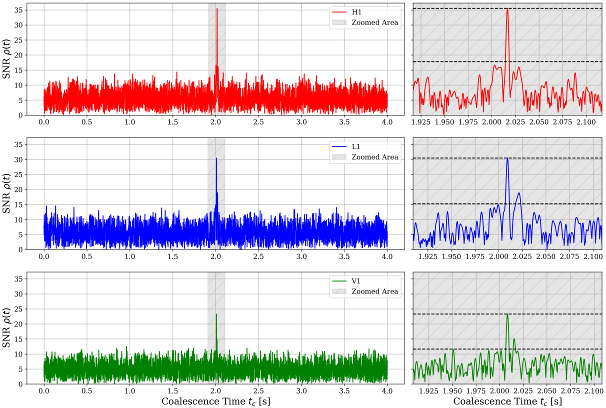
Filter Optimisation:

- GW waveform fixed, but detector response varies
- Depends on antenna patterns $F_+, F_\times(\text{RA}, \text{Dec}, \psi, t_{\text{GPS}})$
- Filter bank: mesh grid over RA, Dec, ψ
- Maximise SNR for each detector

Detector	σ_{SNR}	σ_{t_c} **(s)**	**RA (rad)**	**Dec (rad)**	ψ **(rad)**
Hanford (H1)	5.329	0.001	0.381	0.796	1.988
Livingston (L1)	4.563	0.001	3.998	-0.535	2.885
Virgo (V1)	3.491	0.001	5.775	-0.739	2.372

Table: Standard deviations of SNR and t_c across grid of RA, Dec, and ψ, with parameters that maximise the SNR for each detector.

Matched Filtering: Results



	SNR	Time (s)
H1	35.52	2.017 ± 0.002
L1	30.48	2.009 ± 0.002
V1	23.28	2.008 ± 0.002

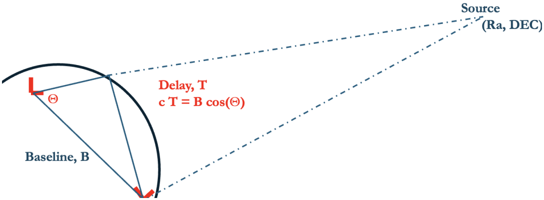
Sky Triangulation: Theory

For a given α, δ - 'true' time-delays from geocenters combined:

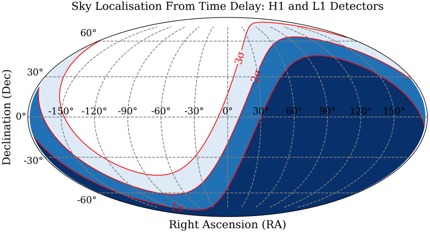
$$\Delta t_{1,2}^{\text{true}}(\alpha, \delta) = \tau_1(\alpha, \delta) - \tau_2(\alpha, \delta). \tag{11}$$

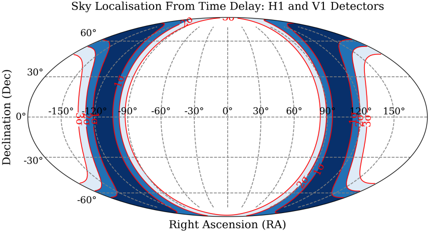
For N baselines - Gaussian Likelihood Model:

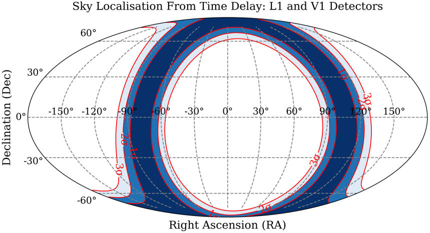
$$\log \mathcal{L}(\alpha, \delta) \propto -\frac{1}{2} \sum_{i<j} \left(\frac{\Delta t_{i,j}^{\text{obs}} - \Delta t_{i,j}^{\text{true}}(\alpha, \delta)}{\sigma_{i,j}} \right)^2 \tag{12}$$

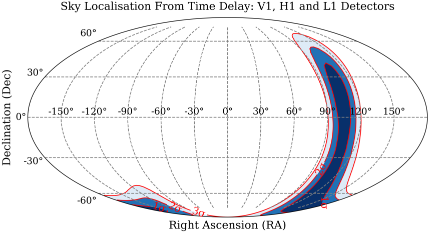


Sky Triangulation: Results









Sky Triangulation: Results

Areas of Confidence Intervals:

Detectors	1σ % Sky	2σ % Sky	3σ % Sky
H1 - L1	38.0	50.7	64.2
H1 - V1	17.3	29.7	43.8
L1 - V1	23.9	38.4	51.8
H1 - L1 - V1	3.9	9.1	15.4

Comparison of Baselines' Lengths:

Detector Pair	Time Delay (ms)	Total Light Travel Time (ms)
H1 - L1	7 ± 3	10
H1 - V1	8 ± 3	27
L1 - V1	0.1 ± 3.0	26

Bayesian Analysis for GW Waves:

Formulation of Likelihood (Noise Only):

For stationary - Gaussian Noise:

$$\mathcal{L}(n \mid H_{\mathrm{noise}}) = \prod_{i=0}^{N} \mathcal{L}(\tilde{n}(f_i) \mid H_{\mathrm{noise}}) = \mathcal{N} \exp\left(-\frac{(n|n)}{2}\right) \tag{13}$$

Inner product is defined using the PSD $S_n(f)$:

$$(a|b) = 4\,\Re\left[\int_0^\infty \mathrm{d}f\, \frac{\tilde{a}(f)\tilde{b}^*(f)}{S_n(f)}\right] = 4\,\Delta f\,\Re\left[\sum_{i=0}^{N_f-1} \frac{\tilde{a}(f)\tilde{b}^*(f)}{S_n(f)}\right] \tag{14}$$

Bayesian Analysis for GW Waves:

Formulation of Likelihood (Signal):

For a given signal with a detector response $h_{\vec{\theta}}(t)$:

$$\mathcal{L}(d, \vec{\theta}) = \mathcal{P}(d \mid \vec{\theta}, H_{\text{signal}}) = \mathcal{P}(d - h_{\vec{\theta}} \mid H_{\text{noise}}) \tag{15}$$

$$\mathcal{L}(d, \vec{\theta}) = \mathcal{N} \exp\left(-\frac{\langle d - h_{\vec{\theta}} | d - h_{\vec{\theta}} \rangle}{2}\right) \tag{16}$$

$$\log \mathcal{L}(d \mid \vec{\theta}) \propto -2\,\Delta f \sum_{i=0}^{N_f - 1} \frac{|\tilde{d}(f_i) - \tilde{h}_{\vec{\theta}}(f_i)|^2}{S_n(f_i)} \tag{17}$$

Bayesian Analysis for GW Waves:

Formulation of Multi Detector Likelihood:

For multiple detectors, $\{d_{\mathcal{I}}\}_{\mathcal{I}=1}^{N_{\mathrm{det}}}$:

$$\mathcal{P}(\{d_{\mathcal{I}}\} \mid \vec{\theta}) = \prod_{\mathcal{I}} \mathcal{P}(d_{\mathcal{I}} \mid \vec{\theta}) \propto \exp\left(-\sum_{\mathcal{I}} \frac{\langle d_{\mathcal{I}} - h_{\vec{\theta}} \mid d_{\mathcal{I}} - h_{\vec{\theta}} \rangle}{2}\right) \tag{18}$$

$$\log \mathcal{L}(\{d_{\mathcal{I}}\} \mid \vec{\theta}) \propto -2\Delta f \sum_{\mathcal{I}=1}^{N_{\mathrm{det}}} \sum_{i=0}^{N_f - 1} \frac{|\tilde{d}_{\mathcal{I}}(f_i) - \tilde{h}_{\vec{\theta},\mathcal{I}}(f_i)|^2}{S_n^{(\mathcal{I})}(f_i)} \tag{19}$$

Adjusting Waveform Model

1. Antenna Response: Adjust for Sky Position

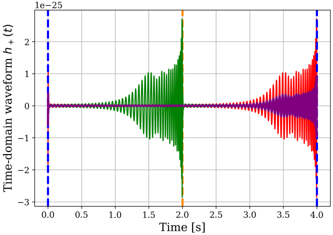
$$\tilde{h}_{\text{det}}(f) = F_+(RA, Dec, \psi)\,\tilde{h}_+(f) + F_\times(RA, Dec, \psi)\,\tilde{h}_\times(f)$$

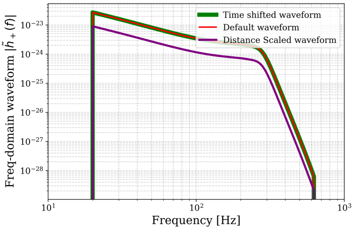
2. Fourier Shift: Shift to Geocentric Coalescence Time

$$t_c^{\text{det}} = t_c^{\text{geo}} + \Delta t_{\text{delay}}(RA, Dec) \quad \Rightarrow \quad h_{\text{det}}(t_c^{\text{geo}}) \xrightarrow{\mathcal{F}} \tilde{h}_{\text{det}}(f)\, e^{2\pi i f t_c^{\text{det}}}$$

3. Scaling: Adjust for Luminosity Distance

$$\tilde{h}_{\text{det}}(f; D_L) = \left(\frac{1\text{ Gpc}}{D_L}\right)\tilde{h}_{\text{det}}(f)$$



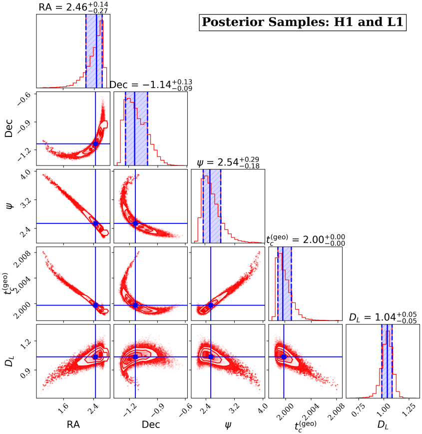


Prior Distributions

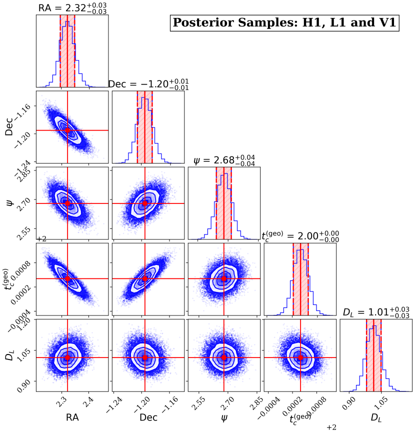
Parameter	**Prior Description**
RA	Uniform over $[0, 2\pi]$
Dec	Uniform in solid angle: $\sin(\text{Dec}) \in [-1, 1]$
ψ	Uniform over $[0, \pi]$
t^c_{geo}	Uniform over $[0, 4]$ seconds
D_l	Log-uniform over $[10^{-2}, 100]$ Gpc (i.e. 10 Mpc to 100 Gpc)

Nested Sampling

Likelihood: H1 and L1

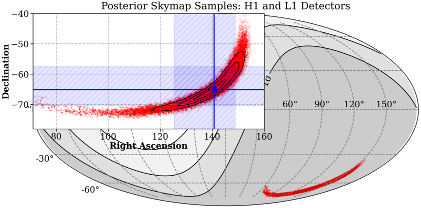


Likelihood: H1, L1 and V1

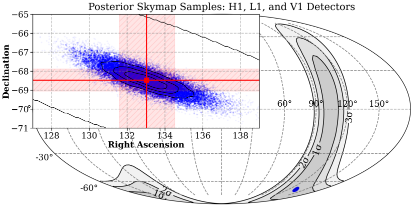


Sky Map Distributions

Skymap: H1 and L1



Skymap: H1, L1 and V1



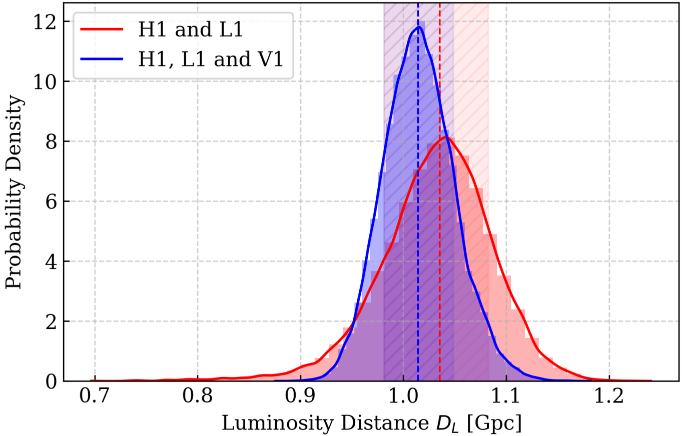
Posterior Localisation:

Detectors	1σ % Sky	2σ % Sky	3σ % Sky
H1 - L1	0.15	0.41	0.81
H1 - L1 - V1	0.00	0.01	0.02

Triangulation Localisation (for context):

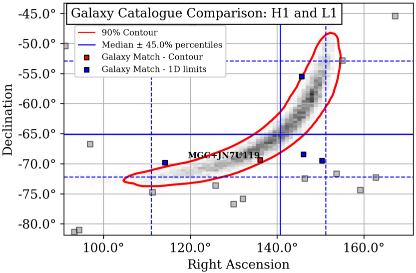
Detectors	1σ % Sky	2σ % Sky	3σ % Sky
H1 - L1	38.0	50.7	64.2
H1 - L1 - V1	3.9	9.1	15.4

Luminosity Distance, D_L

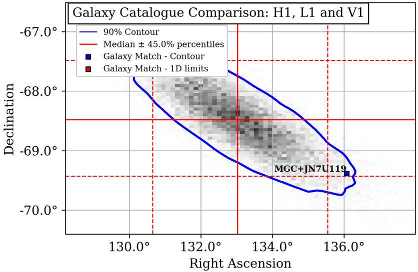


Host Galaxy Identification

Posterior: H1 and L1



Posterior: H1, L1 and V1



Name	RA	Dec	Redshift, z
MGC+JN7U119	2.375	-1.211	0.226

Cosmological Inference: Theory

Hubble Constant, H_0 (Z<<1):

$$H_0 = \frac{c \cdot z}{D_L} \tag{20}$$

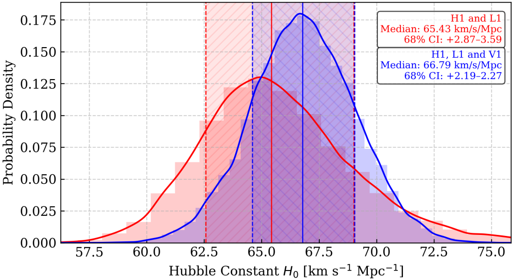
Caveated:

$$D_L = (1+z)\, S_k[r_{CM}] \quad r_{CM} = c\int_0^z \frac{1}{H(z')} dz' \tag{21}$$

$$D_L \approx \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + \cdots\right], \quad q_0 = \frac{1}{2}\Omega_{M,0} - \Omega_{\Lambda,0} \tag{22}$$

Hubble Constant

Posterior Distributions:



Hubble Tension:

Method	H_0 [km/s/Mpc]
Distance ladder (Milky Way Cepheids)	73.2 ± 1.3
H0LiCOW Gravitational lensing	73.3 ± 1.7
Planck CMB	67.4 ± 0.5

Questions

Thank you for your attention!

Jacob Tutt

Department of Physics, University of Cambridge

jlt67@cam.ac.uk

https://github.com/jacobtutt