Determining the Sun's Gravitational Effects on the Stability of the Earth-Moon Lagrange Points

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This paper investigates the gravitational effect of the Sun on the Earth-Moon system through the comparison between the Bi-Circular Restricted Four Body Problem and the Circular Restricted Three Body Problem. It aims to greater understand the true stability of Lagrange points, specifically L3, L4 and L5, when perturbing forces are accounted for. It achieves this by firstly studying the changes in the position of the instantaneous equilibrium points over the 'Sun's orbit', which for L3, located at [-1.0042802, 0.0000000] LD in the Earth-Moon plane varies up to 1.815 LD over the orbital period. Additionally, L4/ L5 points located at [0.4874655, $\pm$0.8653501] are seen to vary by up to 0.9342 LD. The paper then goes on to use the Fourth-Order Runge-Kutta method to numerically integrate the equations of motion and model the position of objects within cis-lunar space over time. It finds the Sun significantly destabilises the Lagrange points, with L3's stable time frame falling from 441.13 to 26.06 days and L4/L5, a previously considered 'stable' point, only remaines in the region for 16.52 days. It goes on to find more stable positions exist within the system, such as the initial instantaneous equilibrium positions. By using E3(0), [-1.0012284,0.0000000] LD, as a starting position a body is able to remain within a stable location for 199.75 days and similarly the use of E5(0), [0.2798798, -0.9567248] LD, results in a maximiun variation of up to 3.1156x10^{-6} LD over 100 years.

1. Introduction

Over the last two decades, space-based technology has rapidly permeated into the civil, commercial and defence sectors, leaving the developed world 'dependent on space to provide economic and social infrastructure' [1], as well as military reassurances. As a result, cis-lunar space, which describes the 'spherical volume that extends outwards from the Earth's geosynchronous region to encapsulate the Moon's orbit and its Lagrange points' [2] (L1-L5), has become increasingly populated. Between the man-made objects, asteroids and meteors, 18,748 objects are currently being tracked by the US Air Force's SST network [3].

There exists a constant drive to reduce the margin of error within orbital mechanics simulations, not only due to the increasingly complex environment but the need to design more advanced missions that span larger time scales. These missions not only have value in themselves, but additionally 'play an important role in enabling and reducing risk for future human missions' [4] to deep space destinations such as Near-Earth Asteroids, Mars or beyond. Of note is NASA's Artemis Programme, aimed at establishing a sustainable human presence surrounding the Moon, including a multi-purpose output, the Lunar Gateway, orbiting the Earth-Moon L2 point. The programme's long-term goal is to pave the way for a crewed mission to Mars [5].

However, within multi-body systems, the equations of motion governed by gravitational forces quickly become chaotic and lack closed-form solutions, an issue tackled by scientists for over 300 years. Consequently, models of systems must be restricted and thus vary from real-world scenarios, for example, the commonly used 'three-body problem' first proposed in 1687 by Newton [6]. Despite simplifications, modeling gravitational systems still requires the extensive use of numerical methods, and therefore calculations often become a compromise between the accuracy required and the computational resources available. As we enter an age of high-performance computing and parallel processing, the advancements in computational capacity will allow higher fidelity modeling by incorporating additional perturbing factors and hence increase accuracy.

Within the field, significant attention is devoted to the Lagrange points, made up of collinear equilibrium points, (L1-L3) as well as the triangular equilibrium points L4 and L5 [7]. The liberation points reside where the gravitational and any fictitious forces balance, demonstrated in Figure 1 by the stationary points of the effective potential in the Earth-Moon rotational plane. This warrants particular interest due to their ability to keep objects in a constant location relative to the Earth and Moon with minimal station keeping and, therefore economically host a range of long-term missions. The Earth-Moon system can be shown to have a sufficient mass ratio (greater than 24.96), such that the Coriolis force acts as a stabiliser [8] around the triangular equilibrium points. As a result, no station-keeping is required over the mission's lifetime, making them ideal locations to host space stations and observatories. On the other hand, the Earth-Moon colinear equilibrium points are, in general, unstable; however, they offer unique advantages such as L2's 'continuous coverage of the dark side of the Moon for communication with future Moon bases' [9]. As a result, there is a significant focus on determining stable periodic or quasi-period orbits around them.

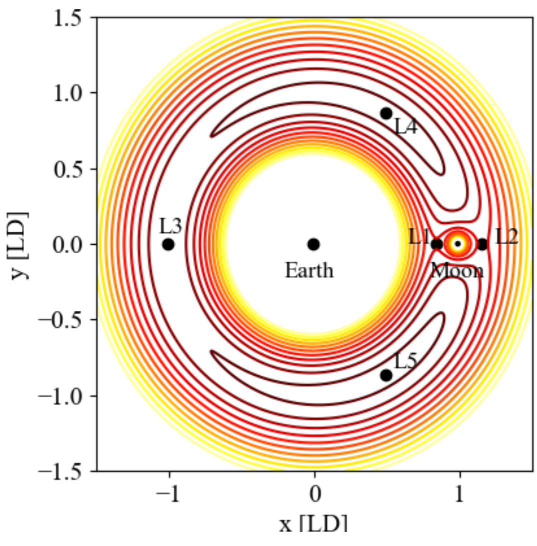

FIG. 1: The effective potential surrounding the rotational plane of the Earth-Moon system defined by the CR3BP. The Lagrange points (L1-L5) are labeled to show the system's stationary points

This paper studies the motion of bodies within the Earth-Moon system (cis-lunar space) and analyses the accuracy of the Circular Restricted Three Body Problem (CR3BP). This is achieved by focusing on the effects of incorporating the Sun via the Bi-circular Restricted 4 Body Problem (BR4BP). The Lagrange points, especially L3, L4, and L5, are used throughout this paper to demonstrate the effect of the Sun on the Earth-Moon system and determine the nature of stable positions within the BR4BP. Firstly, this is investigated through the position of the equilibrium points for all orientations of the system by seeking solutions to the equations of motion. This is extended by determining the stability of these equilibrium points by mapping the future positions of bodies forwards in time through numerical integration. Which

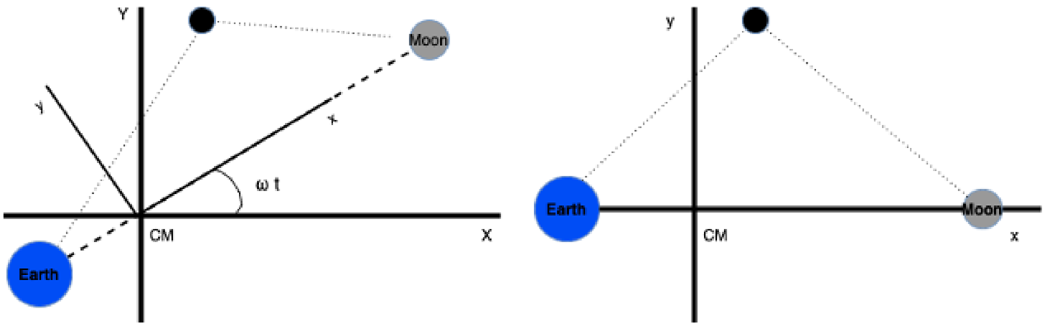

FIG. 2: The two reference frame used to visualise the CR3BP, the inertial frame (left) and the rotational frame of the Earth-Moon System (right)

allows the analysis of whether station keeping is required for triangular equilibrium points when the Sun is accounted for.

In Section 2, the formulation of CR3BP and the BR4BP and the Lagrangian mechanics utilised are discussed. The locations of the instantaneous equilibrium positions over the orientations of the system are described in Section 3. Section 4 evolves bodies' motion over time to understand their stability. Finally, Section 5 discusses the limitations of the results and the implications they have within the field.

2. Lagrangian Mechanics

This paper focuses on the comparison between the CR3BP and the BR4BP [10], which provide representations of the Earth-Moon System and the Sun-Earth-Moon System respectively. The models provide the equations of motions for an additional body (i.e. a spacecraft) of mass M_{sc} within the gravitation system. Despite their individual characteristics, they both follow a common set of restrictions. Firstly, the mass of the additional body is negligible in comparison to that of the primary bodies ($M_{sc} << M_{Sun}, M_{Earth}, M_{Moon}$), and therefore, does not influence their motion. Secondly, the primary bodies are assumed to move in circular orbits about their respective barycentres. And finally, all bodies are taken to lie within the same rotational plane which this paper thus restricts analysis to.

In this paper, we extensively leverage Legendre's transform to convert between reference frames, as outlined in Section 2 A and Section 2 B. This not only helped visualisation over time but vitally improved computational efficiency. By altering the equations of motion, the number of primary bodies varying in position over time was reduced, and hence so was the volume of iterated calculations required. To quantify this benefit, a test body was placed in an arbitrary location in the rotational plane of (0.5, 0.5) Lunar Distances(LD) and evolved forward in time by 1 year using Fourth Order Runge Kutta Methods with a step size of 10 seconds, described in Section 4 A. This was performed in both inertial and rotational reference frames for both the CR3BP and the BR4BP. The CR3BP in the inertial and rotational frames saw a processing time of 186 seconds and 134 seconds respectively, corresponding to a 28% increase in processing speed. Similarly, the BR4BP saw a decrease in processing time from 219 seconds to 152 seconds resulting in a 31% increase in processing speed.

Due to the restrictions imposed, the systems evolution only depends on the masses of the primary bodies and the distances between them. Throughout this paper, we have taken the mass of the Sun (M_{Sun}), Earth (M_{Earth}), and Moon (M_{Moon}) as 1.989×10^{30} kg, 5.972×10^{24} kg and 7.348×10^{22} kg respectively [11]. And the Earth-Moon distance ($|\overrightarrow{r_{EM}}|$) and Sun-Earth distance ($|\overrightarrow{r_{SC_2}}|$) as 3.847×10^{8} m and 1.496×10^{11} m respectively [11].

A. The Circular Restricted 3 Body Problem (CR3BP)

As shown in Figure 2 (left), the Earth and the Moon are modelled in circular orbits acting exclusively under each other's gravitational field and therefore both rotate around a common barycenter at their shared angular velocity, ω, which corresponds to an orbital period of 27.32 days.

$$\omega^2=\frac{G(M_E+M_M)}{|\overrightarrow{r_{EM}}|^3} \qquad \mu=\frac{M_M}{M_E+M_M} \tag{1}$$

By switching from the (X,Y) inertial reference frame to (x,y) rotational reference frame via the coordinate transform shown in Equation (2), the positions of the primary bodies (Earth and Moon) become fixed in time, $\overrightarrow{r_E}$ and $\overrightarrow{r_M}$ respectively. These locations are given in terms of the mass parameter, μ defined in Equation (1), and can be seen in Figure 2 (right).

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} x\cos\omega t - y\sin\omega t \\ y\cos\omega t + x\sin\omega t \end{pmatrix} \tag{2}$$

$$\overrightarrow{r_E}=\begin{pmatrix} -\mu\overrightarrow{r_{EM}} \\ 0 \end{pmatrix} \qquad \overrightarrow{r_M}=\begin{pmatrix} (1-\mu)\overrightarrow{r_{EM}} \\ 0 \end{pmatrix} \tag{3}$$

By utilising Legendre's transform, the equations of motion within the rotational reference frame can be defined, as shown in Equation (4). The acceleration on the body is becomes not only a result of the gravitational force but the fictitious centrifugal and coriolis forces.

$$\ddot{x}=2\omega\dot{y}+\omega^2x-\frac{GM_E(x-x_E)}{|\overrightarrow{r}-\overrightarrow{r_E}|^3}-\frac{GM_M(x-x_M)}{|\overrightarrow{r}-\overrightarrow{r_M}|^3} \tag{4a}$$

$$\ddot{y}=-2\omega\dot{x}+\omega^2y-\frac{GM_E(y-y_E)}{|\overrightarrow{r}-\overrightarrow{r_E}|^3}-\frac{GM_M(y-y_M)}{|\overrightarrow{r}-\overrightarrow{r_M}|^3} \tag{4b}$$

B. The Bi-Circular Restricted 4 Body Problem (BR4BP)

The BR4BP extends beyond the CR3BP and can be described as two coexisting pairs of gravitational systems rotating independently of one another. In the case of the Sun-Earth-Moon system, demonstrated in Figure 3 (left), the two systems can be described as the Earth-Moon system rotating around their centre of mass, C_2 and then the Sun-C_2 system rotating around the centre of mass, C_1. The additional restriction this model must include is that the perturbations induced by the Sun do not influence the Earth-Moon orbit. As a result, the mass parameter, μ_2 and anguluar velocity, ω_2 are equal to those defined in Section 2 A. But the Sun-C_2 system's mass parameter, μ_1 and angular momentum, ω_1 must also be considered and are given in Equation (5).

$$\omega_1^2=\frac{G(M_E+M_M+M_s)}{|\overrightarrow{r_{SC_2}}|^3} \qquad \mu_1=\frac{M_E+M_M}{M_S+M_E+M_M} \tag{5}$$

Inorder to achieve a comparable reference frame to the CR3BP, the coordinate transform translated from the inertial frames centre C_1 to the centre of the Earth-Moon system, C_2 and then performed a rotation defined by the angular velocity, ω_2, outlined in Equation (6). As shown in Figure 3 (right), this achieves a

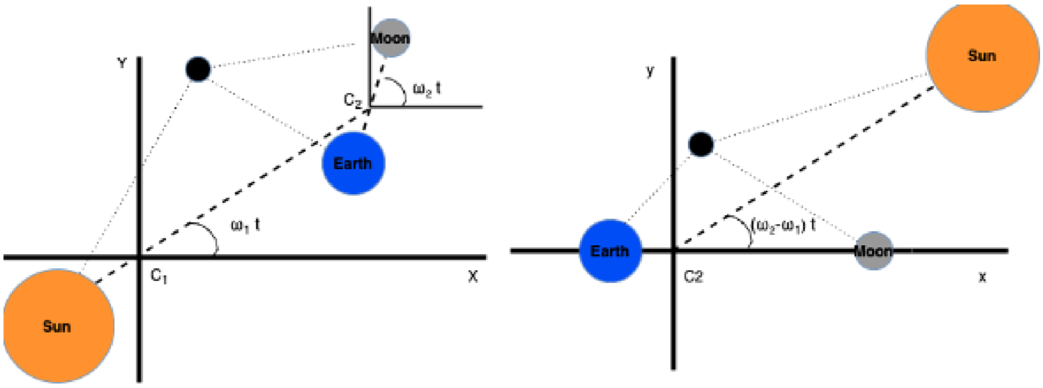

FIG. 3: The two reference frame used to visualise the BR4BP, the inertial frame centered at C_1 (left) and the rotational frame of the Earth-Moon System centered at C_2 with the Sun 'orbiting' (right)

frame in which the Earth and Moon's position are fixed at $\vec{r_E}$ and $\vec{r_M}$ as defined in Equation (3). The Sun then rotates the centre, C_2, at a angular velocity of $\omega_2 - \omega_1$, corresponding to a orbital period of 29.53 days, resulting in the time dependent position $\vec{r_S}$ given in Equation (7).

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} |\overrightarrow{r_{SC_2}}|\cos\omega_1 t + x\cos\omega_2 t - y\sin\omega_2 t \\ |\overrightarrow{r_{SC_2}}|\sin\omega_1 t + y\cos\omega_2 t + x\sin\omega_2 t \end{pmatrix} \quad (6)$$

$$\vec{r_S} = \begin{pmatrix} \overrightarrow{r_{SC_2}}\cos(\omega_2 - \omega_1)t \\ \overrightarrow{r_{SC_2}}\sin(\omega_2 - \omega_1)t \end{pmatrix} \quad (7)$$

As before, Legendre's Equation is used to determine the equations of motion within the rotational frame, which now include the gravitational forces from the Sun, Earth and Moon, as well as the ficticious forces. A non trivial component of the equations to highlight are the final terms which are a result of the translation between barycenters.

$$\ddot{x} = 2\omega_2\dot{y} + {\omega_2}^2 x - \frac{GM_E(x - x_E)}{|\vec{r} - \vec{r_E}|^3} - \frac{GM_M(x - x_M)}{|\vec{r} - \vec{r_M}|^3} - \frac{GM_S(x - x_s)}{|\vec{r} - \vec{r_s}|^3} - \overrightarrow{r_{SC_2}}\omega_1^2\cos(\omega_2 - \omega_1)t \quad (8a)$$

$$\ddot{y} = -2\omega_2\dot{x} + {\omega_2}^2 y - \frac{GM_E(y - y_E)}{|\vec{r} - \vec{r_E}|^3} - \frac{GM_M(y - y_M)}{|\vec{r} - \vec{r_M}|^3} - \frac{GM_S(y - y_s)}{|\vec{r} - \vec{r_s}|^3} - \overrightarrow{r_{SC_2}}\omega_1^2\sin(\omega_2 - \omega_1)t \quad (8b)$$

3. Variation in Equilibrium Positions

A. Methods for Solving Equilibrium Positions

Although the CR3BP produces equilibrium solutions fixed in time (L1-5), in the BR4BP reference frame the Sun's location is time dependent and therefore the corresponding equilibrium points are instantaneous (E1-5). These variations from the original Lagrange points are used to quantify the effects the Sun within cis-lunar space, offer an insight into its effect of the dynamical system and thus represent the importance of its inclusion within orbital modelling.

The instantaneous equilibrium positions refer to where the gravitational and fictitious forces cancel for a given Sun angle ($\theta = (\omega_2 - \omega_1)t$). They are therefore located where a stationary body ($\dot{x} = \dot{y} = 0$), has an acceleration, given in Equation (8), equal to zero in both dimensions ($\ddot{x} = \ddot{y} = 0$). These solutions to the multi-dimensional acceleration were identified using 'multivariate minimisation with Newton's Method' through Python's 'Scipy.Opitimise' library, which utilises the Jacobian matrix as a coefficent matrix [12]. Due to the potential for more than 5 coexisting equilibrium points once the Sun was introduced, the starting conditions were taken as a (1000 x 1000) meshgrid of points within a square ± 1.5 LD from the center C_2. Each starting location was then iterated using Newton's method to attempt to find an solution for the system to within a precision of $\pm 1000m$. Once complete all successful solutions were tested for uniqueness against each other. This was repeated for Sun orientations between 0 and 360° every 0.1°. The method was also used to identify the location of the commonly studied L1-5 points within the CR3BP, using Equation (4), to allow comparison and verification with literature values.

In comparable literature, authors express all equation of motion in their non dimensional form in which masses, distances, angular velocities and hence time are normalised [10]. Although this provides the benefit of generalising the system and reducing the unnecessary constants within the calculations, it was found to in fact hinder the overall accuracy of the results in this paper due to the methods and programming language used, Python. By reducing the units of distance by a factor of 3.847×10^8 (1 lunar distance) and increasing the units of time by a factor of $3.756 \times 10^5, (\frac{1}{\omega_2})$, the overall values of acceleration are reduced by $\sim 1.84 \times 10^{-20}$. This required the level of precision to be significantly higher when numerically solving the equations and thus increased the processing demand beyond the resources available.

B. Results for Solving Equilibrium Positions

Figure 4, shows the spread of instantaneous equilibrium points across a full 'orbit' of the Sun around the Earth-Moon system. Firstly, the accuracy of the numerical methods employed was verified by comparing the positions achieved for the Lagrange points

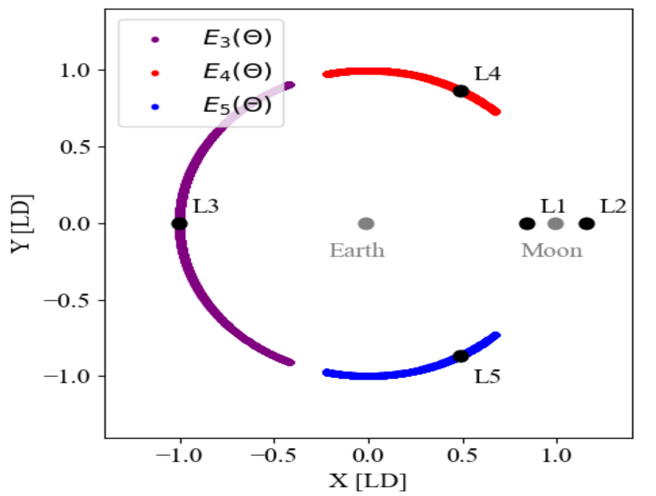

FIG. 4: The spread of instantaneous equilibrium points (E1-5) causes by Sun perturbations over an effective orbit of 29.53 days, in comparison to the CR3BP fixed Lagrange Points (L1-5)

TABLE I: The locations determined using Newton's Method for the Lagrange points (L1-5) within the CR3BP and the total spread in the instantaneous equilibrium points (E1-5) within the BR4BP.

Equilibrium Position	Position in CR3BP $\pm 3\times10^{-6}LD$	Total Variation in BR4BP $\pm 6\times10^{-6}LD$
L1/ E1	$\begin{pmatrix}0.8363624\\0.0000000\end{pmatrix}$	$\begin{pmatrix}\Delta X\\\Delta Y\end{pmatrix}=\begin{pmatrix}0.0013004\\0.0035483\end{pmatrix}$
L2/ E2	$\begin{pmatrix}1.1547940\\0.0000000\end{pmatrix}$	$\begin{pmatrix}\Delta X\\\Delta Y\end{pmatrix}=\begin{pmatrix}0.0027276\\0.0091717\end{pmatrix}$
L3/ E3	$\begin{pmatrix}-1.0042802\\0.0000000\end{pmatrix}$	$\begin{pmatrix}\Delta X\\\Delta Y\end{pmatrix}=\begin{pmatrix}0.5880702\\1.8150390\end{pmatrix}$
L4/ E4	$\begin{pmatrix}0.4874655\\0.8653501\end{pmatrix}$	$\begin{pmatrix}\Delta X\\\Delta Y\end{pmatrix}=\begin{pmatrix}0.8960414\\0.2673181\end{pmatrix}$
L5/ E5	$\begin{pmatrix}0.4874655\\-0.8653501\end{pmatrix}$	$\begin{pmatrix}\Delta X\\\Delta Y\end{pmatrix}=\begin{pmatrix}0.8960414\\0.2673181\end{pmatrix}$

within the CR3BP, shown in Table I, with literature data [10]. They were all shown to agree to within $\pm 0.00005LD$ ($\pm 19235m$). Due to this very strong correlation, we could confirm our methods worked and further results were valid. However, the error was an order of magnitude greater than the precision the program sought ($\pm 1000m$), which was likely a result of slight differences in the systems input parameters such as mass and separation. Thus demonstrating the additional sources of uncertainty that must be considered other than those from the computational methods.

By comparing the extent to which the Sun perturbs the (instantaneous) equilibrium positions (E1-5), it was found that the maximum variations in E1 and E2 are more than two orders of magnitude smaller than that of E3, E4, E5, (shown in Table I) and thus are not shown within Figure 4 as their deviations from L1 and L2 could not be seen. This is as expected due to their proximity to the Moon and therefore the reduced significance of the Sun's influence. Secondly due to their symmetrical locations, the variations in E4 and E5 over the Sun orbit are found to be exact inverses of one another, and consequently although E5/L5 is focused on within the paper, the results for E4/L4 are directly comparable.

It is easy to assume that the Sun's presence has the greatest overall effect on E3, with its maximal positional spread being twice that of E5 (in opposite directions), however when considering stability it is also important to consider the rate of change of these variations. From Figure 5, it is clear that the maximal equilibrium shift of E5 (in both x and y directions) happens at Sun orientations

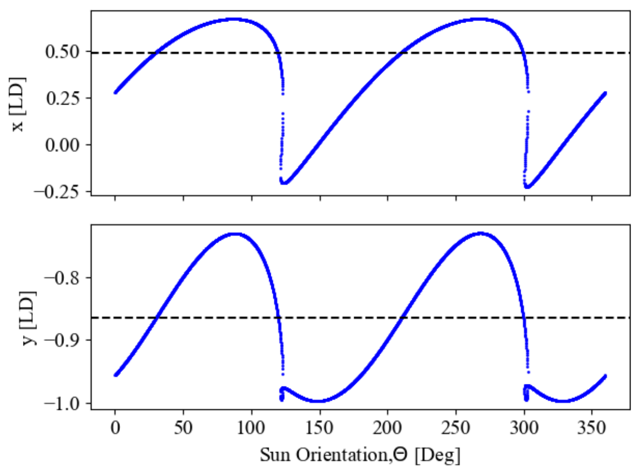

FIG. 5: The positional variation of E5 over the Sun's 'orbit' of the Earth-Moon Sytem (360°) along the x axis (top) and y axis (bottom). The fixed location of the corresponding Lagrange point, L5 is also marked by dashed line.

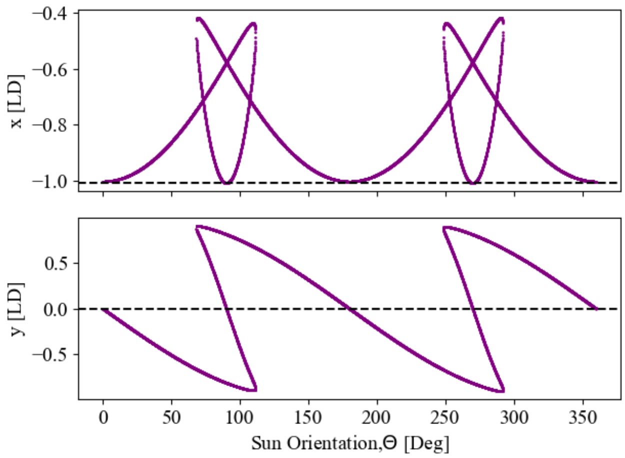

FIG. 6: The positional variation of E3 over the Sun's 'orbit' of the Earth-Moon Sytem (360°) along the x axis (top) and y axis (bottom). The fixed location of the corresponding Lagrange point, L3 is also marked by dashed line.

of $\sim 120°$ and $\sim 300°$. And occurs within a (Sun) phase change of 32.53° corresponding to a time frame of 2.67 days. Therefore this shows a sudden and rapid shift in the dynamical system which could significantly effect the stability of the L4 and L5 points.

On the other hand, the variation of E3, shown in Figure 6, shows a much more convoluted trend, due to the break down of the one to one relationship. This indicates the presence of multiple (up to 3) instantaneous equilibrium points within the region of L3, for Sun orientations between 68.19 and 111.41° as well as 148.19 and 291.41°. And therefore although the maximum difference in equilibrium position is large, shown in Table I, this occurs between 'separate points' and thus can not be taken as an accurate measure of the regions dynamical volatility .

4. Effect on Dynamical Stability

In order to determine the dynamical consequences of these shifts in equilibrium points over time, numerical integration was employed to model bodies' motion forwards in time. This allowed us to determination whether the Coriolis force surrounding L4 and L5 acts as a sufficient natural counter to the Sun's perturbations and thus whether they can still be considered 'stable', as well as quantify the effects on the already unstable L3 point.

A. Methods for Investigating Dynamical Stability

The differential equations of motion for the gravitational systems, given within Equation (4) and Equation (8), were solved using the fourth-order Runge-Kutta (RK4) method. Both the CR3BP and BR4BP were analysed inorder to provide a comparison. RK4 was chosen due the benefits in accuracy it offers from taking weighted averages across 4 positions for each time step.

As a standard across the paper, time steps of 100 seconds were used. The motion was modelled 100 years into the future which acts as an upper value for a typical mission's 'life span between 30 and 50 years' [13]. Additionally due to the Sun's time dependent position within the BR4BP and its effect on the equations of motion, an initial Sun orientation of 0° was used across all simulations.

The stability of points was quantified via two methods, firstly stable positions which remain relatively stationary in time, such as L4 and L5 within the CR3BP, where analysed based on their maximal displacement from the original location over the 100 year period. Unstable locations, which drift from their starting position were assessed via the time taken for them to drift 0.2 LD from their original location.

When analysing the stability of 'equilibrium' positions within the BR4BP, both the CR3BP's Lagrange points, L3-5, and the pre-

TABLE II: The positions of the instantaneous equilibrium points, E1-5, at Sun orientations of 0°

	E3(0)	E4(0)	E5(0)
Position $\pm 3\times10^{-6} LD$	$\begin{pmatrix}-1.0012284\\0.0000000\end{pmatrix}$	$\begin{pmatrix}0.2798798\\0.9567248\end{pmatrix}$	$\begin{pmatrix}0.2798798\\-0.9567248\end{pmatrix}$

viously determined instantaneous equilibrium positions, E3-5(θ) at 0° were used as starting positions. These values, seen in Table II, were determined in Section 3 and especially in the case of E4 and E5 vary significantly from their corresponding Lagrange points.

B. Results of Dynamical Stability

The Sun's effects on the L3 point is discussed first. Due to the equilibrium point being unstable, it's relative stability is assessed based on the time frame it remains within 0.2 LD of the original location seen, in Table III.

TABLE III: The stability time frame of the Lagrange point, L3, and its corresponding instantaneous equilibrium position E3(0) within the CR3BP and BR4BP systems

Initial Position	L3	L3	E3(0)
Model	CR3BP	BR4BP	BR4BP
Stable Time Period [Days]	441.13	26.06	199.75

By comparing the behaviour of bodies originating at the L3 point in both dynamical models, it is obvious that the Sun significantly reduces the locations stability as the time period in which the body remains within the region decreases by more than a factor of 16. This suggests that spacecrafts held at the L3 point of the Earth-Moon system would require significantly more station keeping than originally suggested by the CR3BP and thus decreasing their potential lifespan. However, by initially positioning the body at the BR4BP's instantaneous equiluibrium position, which in the case of this simulation starting at 0° is E3(0), we find a much more stable location accounting for the Sun's presence. Here the 'stable time period' is only half that of L3 within the CR3BP, and 4.6 times longer than the L3 point within the BR4BP.

While looking at the effect of the Sun on the stability of the L4 and L5 points, we will once again only discuss the case of the L5 point due to their almost identical results. Within the case of a stable point such as 'L5', the stability is quantified based on the maximum deviation from the starting equilibrium position, as shown in Table IV.

As expected due to the Coriolis force acting as a restoring force, variations in a body's position when placed at L5 within the CR3BP are incredibly small, shown in Table IV, corresponding to a maximum deviation of just 6.89 m. However, we find that if we place a body at L5 within the BR4BP it no longer becomes stable and drifts further than 1000 LD away over the 100 year period. Additionally it drifts outside 0.2 LD of its original location within 16.52 days and therefore is in fact 'less stable' that L3 within the BR4BP model. Similarly to before, the instantaneous equilibrium point E5(0) was then investigated and found to be stable position

TABLE IV: The stability of the Lagrange point, L5, and its corresponding instantaneous equilibrium position E5(0) within the CR3BP and BR4BP systems

Initial Position	L5	L5	E5(0)
Model	CR3BP	BR4BP	BR4BP
Deviation in Position [LD]	1.7926 x 10^{-8}	$>$ 1 x 10^{3}	3.1156 x 10^{-6}

within the BR4BP, only drifting by 1074.26m over the 100 year period. Although this is still a significant jump from the CR3BP, it is a very small distance relative to the scale of cis-lunar space.

Overall the behaviour surrounding the L3 and L5 points, shows that once the Sun is accounted through the BR4BP, there is a significant effect on the stability on the equilibrium points of the CR3BP and therefore its inclusion within orbital modelling is vital. By going on to investigate the behaviour on bodies originating at the instantaneous equilibrium points E3(0) and E5(0) we observe an increase in stability compared to those at the Lagrange points, suggesting them as more useful locations to host spacecrafts such as orbital outposts and observatories.

5. Future Research

A. Implications for Mission Planning

One interesting comparison between the two methods used to study the dynamic stability of cis-lunar space was the the significant difference between the variation in position of equilibrium points and the real world effect this had on the motion of bodies within the system. Using the instantaneous equilibrium position E5 as a demonstration, the results of Section 3 showed huge variations in its position during the Sun's 'orbit' of the Earth and Moon, with a maximium deviation of 0.9342 LD from its initial position. However despite this when a body is placed at E5(0) and modelled forward in time, as done in Section 4, its position only varied by 3.1156 x 10^{-6} LD over 100 years ($\sim$ 1236 'Sun orbits'). This can be attributed to two factors, firstly the Coriolis force being sufficiently large to counter the changes in the dynamic system and secondly the Sun's relatively short orbital period, causing the force to constantly change direction and limit significant linear acceleration on a body.

By further considering the idea that the periodic motion of the Sun causes periodic accelerations within the dynamical system that cancel each other over time, it raises the potential for more stable positions to exist than those that originate at instantaneous equilibrium positions. With significantly higher computational resources and introducing parallel processing techniques, a (1000x1000) meshgrid of points could be numerically integrating forward in time to determine whether such positions existed. One of the major restrictions placed on the simulations within this paper was limiting the initial Sun orientation to exclusively 0 °. In the future one could not only robustly investigate all starting positions within cis lunar space through numerical integration but also all initial Sun orientations. This would not only allow us to determine the most stable starting position but whether certain starting orientations/ times are more stable than others.

One of the assumptions made within this paper that must be reviewed is the feasibility of being able to not only place spacecrafts within precise locations in cis-lunar space, but simultaneously being able to ensure they are stationary within the Earth-Moon rotational frame when doing so. Due to the likely error in mission trajectories to target locations, the effect of variations in the initial parameters such a $\dot{x}$ and $\dot{y}$ have on stability requires further investigation in future research. Alternatively, significant progress has made using invariant manifolds and nonlinear dynamical systems theory to identify stable motion around legrange points such as those in halo or Lyapunov orbits [14].

B. Limitations of BR4BP and Higher Fidelity Modeling

As mentioned in Section 1, modeling the chaotic dynamics of gravitational systems is a very computationally demanding process and therefore models such as the CR3BP and the BR4BP are employed to introduce restrictions to the mechanics of the system. When using these models it is vital to understand the significance of their simplifications and the effect they have on the uncertainty in the results.

Firstly, although the Earth's orbit around the Sun is almost perfectly circular, with an eccentricity of 0.0017, the Moon's orbit of Earth is much more elliptical, with an eccentricity of 0.055, thus deviating from the Bi-Circular model used within this paper [15]. Secondly, the Moon's orbit has an inclination of 5.1° in relation to the ecliptic plane [15] rather than all motion occurring within the same plane as suggested by this paper's analysis. Finally, in addition to all of this, one of the major computational benefits of the Bi-Circular Restricted Four-Body problem is being able to treat the Sun-Earth and Earth-Moon Orbits as gravitationally independent on one another however in reality the Sun causes major perturbations to the Moon's orbit such a shifts in inclination, eccentricity and semi major axis [16].

As a result of the model's deviations from the true Sun-Earth-Moon system, the levels of uncertainty are much higher levels than those of created by precision level of the numerical methods employed. Looking forward, as processing power grows exponentially, the scientific community will continuously be able to improve the accuracy of the simulations used and account for additional factors such as 'photogravitational forces, variation of masses, the Pointing-Robertson and Yarkovsky effects and the atmospheric drag and solar wind' [17].

6. Conclusion

This paper has demonstrated the importance of including the Sun within orbital mechanics simulations of cis-lunar space due to the effects it has on the dynamical system and reducing it's stability. It has firstly done so by determining the spread in instantaneous equilibrium points, in the BR4BP, over the Suns orbit. This provides evidence for the volatility of the gravitational system, with the L3 point, located at [-1.0042802, 0.0000000] LD in the Earth-Moon plane varying up to 1.815 LD over the orbital period. Additionally the L4/ L5 points located at [0.4874655, $\pm$0.8653501] are seen to vary by up to 0.9342 LD.

The effect of these periodic shifts on bodies held at equilibrium points was then assessed. Bodies held at the L3 point were seen to remain within $\pm$0.2 LD from their initial location for only 26.6 days compared to the 441.13 days when the Sun is not accounted for, representing a significant destabilisation of the region. Additionally, those held at the L5 points went from remaining within 1.7926 x 10^{-8} LD of where initially placed for 100 years, to rapidly drifting 0.2 LD within 16.52 days, and therefore could no longer be considered a 'stable' equilibrium position.

It also offers evidence for the existence of more stable positions than those traditionally defined as the Lagrange points from the Circular Restricted Three Body Problem and ultimately better locations for hosting missions. By trialing the instantaneous equilibrium position at 0 $^{\circ}$ as starting locations. The use of E3(0), [-1.0012284,0.0000000] LD, sees a 'stable period' of 199.75 days which is longer than that for the traditional L3 points. And finally the use of E5(0), [0.2798798, -0.9567248] LD, can be seen at a 'stable' point once again, varying by only 3.1156x10^{-6} LD over 100 years, with comparable results for E4(0).

References

[1] M. N. Sweeting, "Modern small satellites-changing the economics of space," vol. 106, no. 3, pp. 343–361, (2018). [Online]. Available: https://doi.org/10.1109/JPROC.2018.2806218

[2] M. Byers and A. Boley, "Cis-lunar space and the security dilemma," vol. 78, no. 1, pp. 17–21, (2022). [Online]. Available: https://doi.org/10.1080/00963402.2021.2014231

[3] "Challenges and potential in space domain awareness," vol. 41, no. 1, pp. 15–18, (2018). [Online]. Available: https://doi.org/10.2514/1.G003483

[4] M. L. L. Bobskill, Marianne R., "The role of cis-lunar space in future global space exploration," (2012). [Online]. Available: https://ntrs.nasa.gov/citations/20120009459

[5] M. Smith, D. Craig, N. Herrmann, E. Mahoney, J. Krezel, N. McIntyre, and K. Goodliff, "The artemis program: An overview of NASA's activities to return humans to the moon," in 2020 IEEE Aerospace Conference. IEEE, (2020), pp. 1–10. [Online]. Available: https://doi.org/10.1109/AERO47225.2020.9172323

[6] Z. E. Musielak and B. Quarles, "The three-body problem," vol. 77, no. 6, p. 065901, (2014). [Online]. Available: https://doi.org/10.1088/0034-4885/77/6/065901

[7] B. Sicardy, "Stability of the triangular lagrange points beyond gascheau's value," vol. 107, no. 1, pp. 145–155, (2010). [Online]. Available: https://doi.org/10.1007/s10569-010-9259-5

[8] J. Singh and B. Ishwar, "Effect of perturbations on the stability of triangular points. in the restricted problem of three bodies with variable mass," vol. 35, no. 3, pp. 201–207, (1985). [Online]. Available: https://url.org/10.1007/BF01227652

[9] J. Lü, Q. Wang, and S. Wang, "Analytic approach on geometric structure of invariant manifolds of the collinear lagrange points," vol. 55, no. 9, pp. 1703–1712, (2012). [Online]. Available: https://doi.org/10.1007/s11433-012-4810-x

[10] A. Tselousova, S. Trofimov, and M. Shirobokov, "Geometrical tools for the systematic design of low-energy transfers in the earth-moon-sun system." 2020 AAS/AIAA Astrodynamics Specialist Conference, (2020). [Online]. Available: https://www.researchgate.net/publication/348001816_Geometrical_Tools_for_the_Systematic_Design_of_Low-energy_Transfers_in_the_Earth-Moon-Sun_System

[11] S. Scheuerle Jr., "Construction of ballistic lunar transfers in the earth-moon-sun system," p. 11715014 Bytes, (2021). [Online]. Available: https://doi.org/10.25394/PGS.14456391.V1

[12] H. Homeier, "A modified newton method with cubic convergence: the multivariate case," vol. 169, no. 1, pp. 161–169, (2004). [Online]. Available: https://doi.org/10.1016/j.cam.2003.12.041

[13] G. Gargioni, D. Alexandre, M. Peterson, and K. Schroeder, "Multiple asteroid retrieval mission from lunar orbital platform-gateway using reusable spacecrafts," in 2019 IEEE Aerospace Conference. IEEE, (2019), pp. 1–13. [Online]. Available: https://doi.org/10.1109/AERO.2019.8741985

[14] H. Baoyin and C. R. McInnes, "Trajectories to and from the lagrange points and the primary body surfaces," Journal of Guidance, Control, and Dynamics, vol. 29, no. 4, pp. 998–1003, (2006). [Online]. Available: https://doi.org/10.2514/1.17757

[15] M. C. Gutzwiller, "Moon-earth-sun: The oldest three-body problem," Rev. Mod. Phys., vol. 70, pp. 589–639, Apr (1998). [Online]. Available: https://doi.org/10.1103/RevModPhys.70.589

[16] F. Espenak and J. Meeus, Five Millennium Catalog of Solar Eclipses: -999 to +3000 (2000 BCE to 3000 CE), ser. NASA technical paper. National Aeronautics and Space Flight Administration, ISBN: 9781941983379 (2008). [Online]. Available: https://books.google.co.uk/books?id=v858twAACAAJ

[17] E. I. Abouelmagd and J. L. Guirao, "On the perturbed restricted three-body problem," vol. 1, no. 1, pp. 123–144, (2016). [Online]. Available: https://doi.org/10.21042/AMNS.2016.1.00010

Scientific Summary for a General Audience

During the 21st century, the developed world has grown increasingly dependent on systems within space. For example, society relies on satellites for everyday tasks such as weather forecasting, bank transfers and navigation with GPS. Looking forward to future missions, such as NASA's Artemis Programme, there is currently significant interest in advancing the human presence in space with projects like the Lunar Gateway, which will support the sustainable exploration of the moon and pave the way for missions to Mars and deep into the solar system. When planning such missions, the Lagrange points, positions in space where gravitational forces cancel with one another, appeal to the scientific community due to their potential to host space stations and observatories in a fixed position with minimal fuel usage.

With the desire to plan ever more complex missions comes the need to be able to model the motion of spacecraft with increasing accuracy. However, the complexity of the gravitational systems in space means that precisely predicting the future motion of bodies requires a vast amount of computational power and therefore, simplifications are often made, and numerical methods are used. This paper analyses the effect of the Sun on bodies within cis-lunar space (the region close to the Earth-Moon system) and demonstrates the importance of including it when performing calculations. It goes on to identify more stable locations in space than those calculated when only considering the Earth and Moon (Lagrange points), which would allow long-term missions to be more economically viable and reduce the need for station keeping.