Mapping Rockets' Trajectories In Gravitational Systems

Ever since Newton first approached the problem in 1687, physicists have tried to understand the differential equations determining the subsequent motion of multiple bodies under their mutual gravitational attraction [1] With the rise in artificial satellites orbiting our solar system and interest in exploiting asteroids due to their potential as 'vast reservoirs of valuable resources' [2], it is growing ever more critical to be able to predict the chaotic motion of bodies entering complicated gravitational systems.

The Restricted Three-Body Problem

Mathematically, trying to map the motion of three bodies within a mutual gravitational field 'involves 18 first-order differential equations' [1], and therefore is commonly simplified to give what is referred to as 'the restricted three-body problem'.

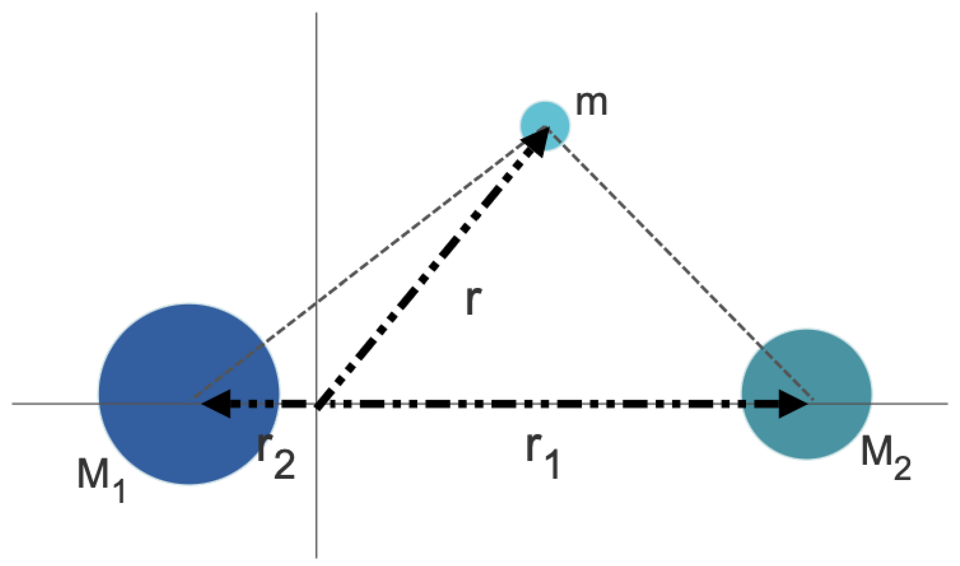

Figure 1. A figure showing the structure of the restricted three-body problem and vector conventions.

This describes two bodies initially at positions, r_1 and r_2, and of significant mass, M_1 and M_2, respectively, rotating about their shared centre of mass in circular orbits, allowing the determination of a third body's subsequent motion within the plane of the original two bodies. The third body must have insignificant mass, m, so it has a negligible effect on the original bodies within the system.

Newton's laws of gravitation dictate that the force exerted on the third body at a position r from the centre of mass is given by:

$$F = -\frac{GM_1 m}{|\vec{r} - \vec{r_1}|^3}(\vec{r} - \vec{r_1}) - \frac{GM_2 m}{|\vec{r} - \vec{r_2}|^3}(\vec{r} - \vec{r_2}) \qquad (1)$$

Due to the bodies' orbits around their centre of mass at a shared angular frequency, w, given by Kepler's third law, r_1 and r_2 are time dependent. However, by using a rotational transform (Eqn 2), the bodies' motion can be observed within a reference frame, in which they are stationary and therefore r_1 and r_2 are constant, using Lagrange's equation (Eqn 3).

$$x'(t) = x\cos\omega t - y\sin\omega t \qquad (2a)$$
$$y'(t) = y\cos\omega t + x\sin\omega t \qquad (2b)$$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{q}_j}\right) - \frac{dL}{dq_j} = 0 \qquad (3)$$

Where L is the Lagrangian.

This creates the equations of motion for the rotational frame shown in Eqn 4, which include inertial forces: the Coriolis Force and the Centrifugal Force.

$$\ddot{x} = 2\omega\dot{y} + \omega^2 x - (1-\mu)\frac{\omega^2(x - x_1)}{|\vec{r} - \vec{r_1}|^3} - \mu\frac{\omega^2(x - x_2)}{|\vec{r} - \vec{r_2}|^3} \qquad (4a)$$
$$\ddot{y} = -2\omega\dot{x} + \omega^2 y - (1-\mu)\frac{\omega^2(y - y_1)}{|\vec{r} - \vec{r_1}|^3} - \mu\frac{\omega^2(y - y_2)}{|\vec{r} - \vec{r_2}|^3} \qquad (4b)$$
$$\mu = \frac{M_2}{M_1 + M_2} \qquad (4c)$$

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Legrange Points

The resulting equations of motion achieved cannot be solved analytically and show chaotic behaviour for many starting locations and velocities. Mathematicians Euler and Lagrange discovered that 'there are five equilibrium points in the vicinity of the two orbiting masses' [3], referred to as Lagrange points.

The equilibrium position results from the gravitational attraction of both bodies exactly counterbalancing the centrifugal force[4]. The chaotic behaviour shown outside of these initial equilibrium positions makes it even more important for physicists to be able to 'find regions that display predictable behaviour' [1].

From Fig.2, we can see that the L_{1-3} lie along the axis connecting the two rotating bodies in the rotational reference frame. And therefore, are given by the solutions of:

$$x - \frac{(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{\mu(x-1+\mu)}{|x+\mu|^3} = 0 \qquad (5)$$

Upon analysis of the stability of the given Lagrange points, the curvature of the effective potential exposes L_{1-3} as dynamically unstable due to them being saddle points, meaning small variations from the exact equilibrium positions will grow exponentially over time [3]. Whereas L_4 and L_5 both show stability due to a correctional Coriolis force effect, meaning they have the potential to create stable orbits of asteroids such as those found on Jupiter's orbits.

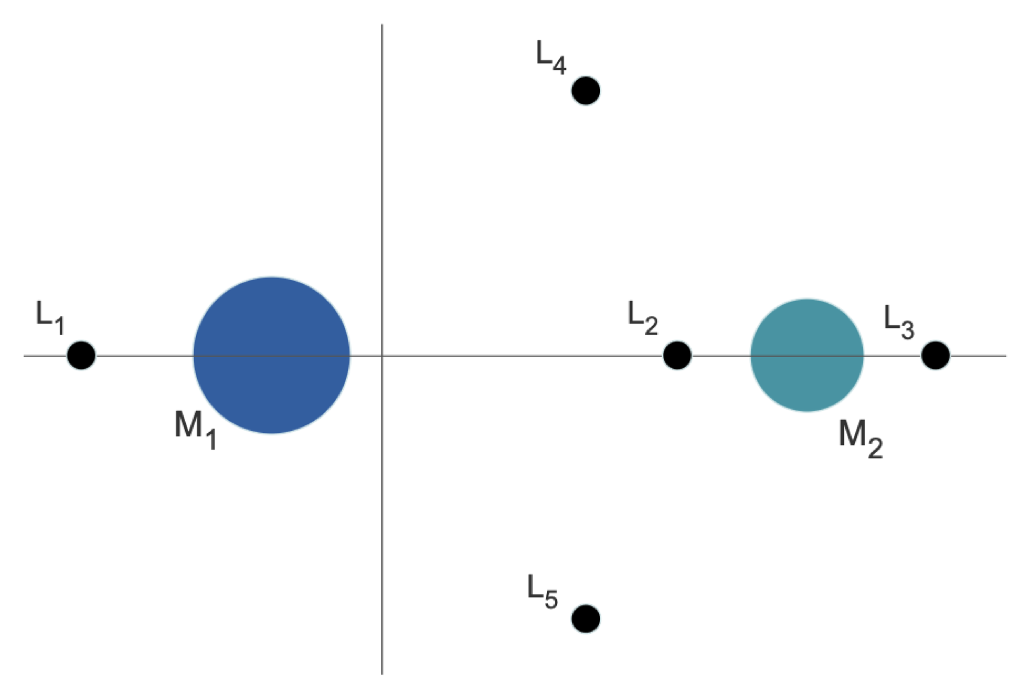

Figure 2. A figure showing the location of Lagrange points L 1-5 relative to the location of the rotating bodies.

Computational Advantage Of Rotating Reference Frames

By building the centrifugal and coriolis force into the differential equation of motions, it has allowed the earth and moon to be stationary in the reference frame and therefore has reduced the requirement to calculate their position on their fixed orbit for each given time step.

This has significantly increased the computational efficiency of the approach and in future will allow for deeper numerical analysis, such as determining a contour plot for the effective potential surrounding the Lagrange points in the rotating reference frame, and thus the stability of Lagrange points.

Numerical Approach

We have applied two standard methods to solve the differential equations outlined in Eqn 3: the Taylor expansion and the fourth-order Runge-Kutta formula. Focusing on the Earth-Moon system, we have mapped the future orbits of low-mass objects within the system's gravitational field.

We have focused our analysis on the area surrounding the L_2 point and to be initially stationary in the rotational frame. Our L_2 point's position was found through solving Eqn 5 using iterative methods and achieved a result of 1.1557 Lunar distances. The points on either side are given by varying the separation from the moon by ± 10% and 20%

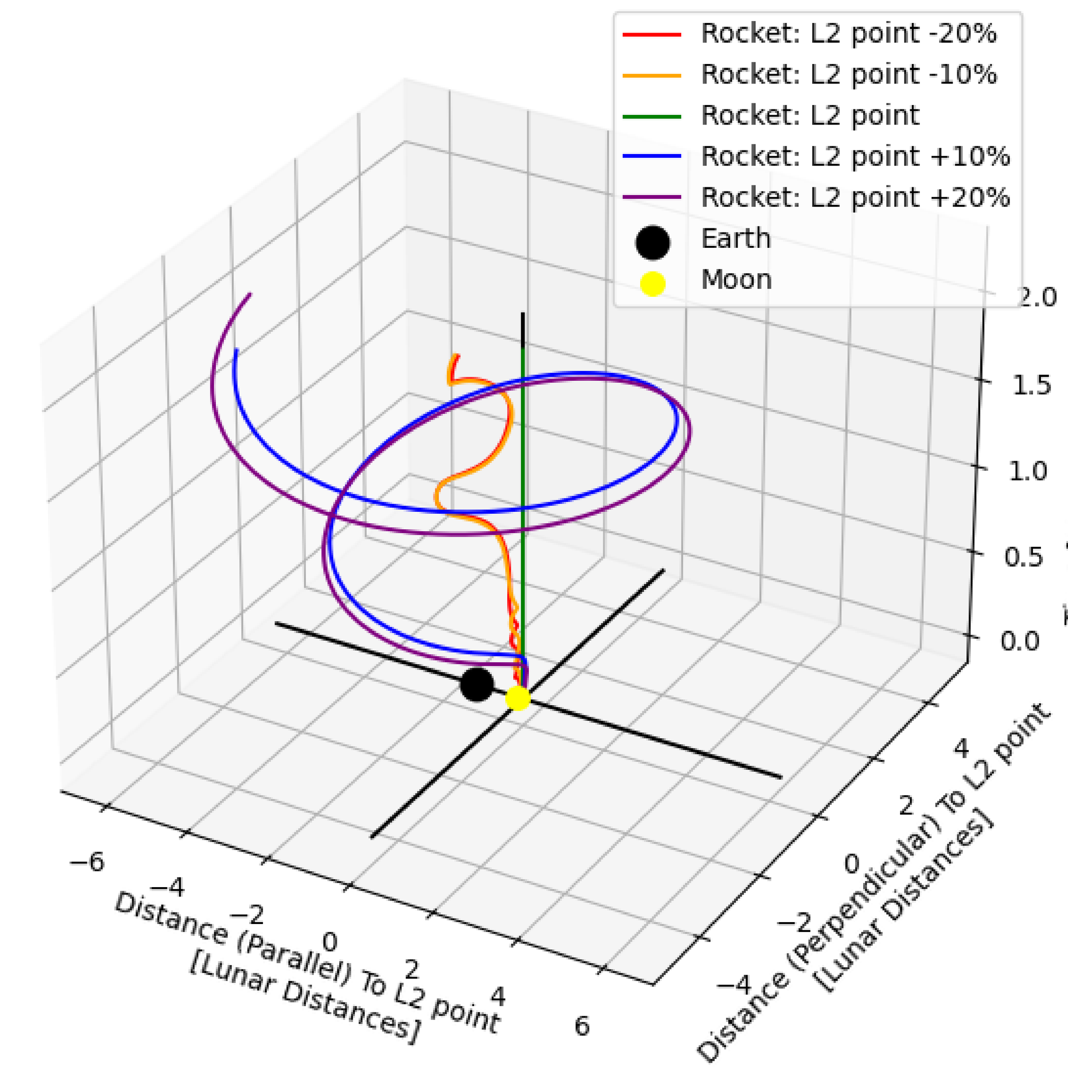

Figure 3. A figure showing the 2D deviations from the L_2 point over 2 orbital periods depending on the initial starting conditions, evaluated using fourth order Runge-Kutta.

Results and Discussion

The future trajectory of the low-mass objects was evaluated using 1,000,000 iterations over the 2 orbital periods resulting in a time step of 4.7140 s.

Analysing the behaviour modelled at the L_2 point itself strongly verifies the numerical approximation used as it only sees a maximum deviation of $1.3417\text{x}10^{-15}$ lunar distances from its initial position over 2 orbital periods, a $8.6172\text{x}10^{-13}$ % change from it initial displacement from the moon.

Fig 3 and 4 both show data calculated from the fourth-order Runge-Kutta method, which is typically a more accurate method for a given time step as a result of taking a weighted average over 4 points around each step.

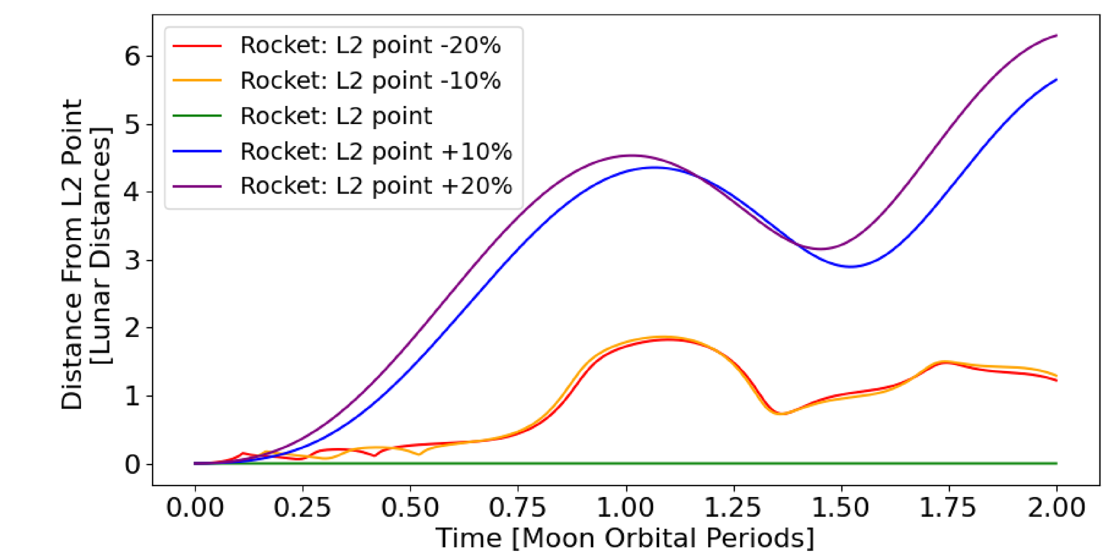

Figure 4. A figure showing the deviation in distance from the L_2 point over 2 orbital periods depending on initial starting conditions, evalutated using fourth order Runge-Kutta.

Fig 4 supports the idea that L_2 is an unstable saddle point. With those trajectories starting closer to the centre of mass quickly falling into an accelerating orbit towards it, therefore explaining why their deviations tend to a constant. And those trajectories starting further from the centre of mass drift outwards as there is insufficient gravitational attraction to balance the centrifugal and coriolis force created by the rotation and therefore 'spinning outwards' from the stable orbit.

Looking Forward

- **Determine Stability Of Legrange Points** and be able to plot contour plots of effective potential.
- **Investigate Trojan Points Within The Sun-Jupiter System** including the stability of their asteroids, and the limits on the initial trajectories that would allow them to form.

References

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